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STJEPAN GRADIĆ ON GALILEO'S PARADOX OF THE BOWL

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ABSTRACT: In his treatise *De loco Galilaei quo punctum lineae aequale pronuntiat* (Amsterdam, 1680), written in 1661, Gradić presented a study and evaluation of Galileo's paradox of the bowl. He accomplished the following: (1) he implicitly disputed the concept of the indivisible; (2) he introduced a new procedure which he designated as *uniformis processio* and contributed to the understanding of the limit in comparison with Galileo's viewpoints; (3) he researched Luca Valerio's proof for the measurement of volume with a curved limit; and (4) he developed a series of topological ideas including a rigorous mathe-matical description of inscribed denticulated forms. As Gradić's professor Bonaventura Cavalieri and Gradić's correspondent HonorP Fabri used the method of indivisibles in their interpretations of Galileo's paradox, they had no influence upon Gradić when he chose his original approach to the problem of the limit of geometrical quantity.

It was in Amsterdam in 1680 that Stjepan Gradić, as prefect of the Vatican Library, printed his physico-mathematical collection which included the treat-

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ise *De loco Galilaei quo punctum lineae aequale pronuntiat*.[1] This was the sole mathematical treatise Gradić ever published (Fig. 1) and has become the subject of recent research.[2]

The first and most unavoidable characteristic of Gradić's research was that, among Galileo's numerous viewpoints, he decided to examine Galileo's dispute on the nature of the infinite (*de infiniti natura*), published in *Discorsi* in 1638. Second, the genesis and publication of Gradić's treatise preceded the appearance of Newton's *Principia* (1687), i. e. the establishment of infinitesimal calculus in the form of the method of the first and last ratios. Therefore, Gradić's consideration about the nature of the infinite belongs to the period determined by two fundamental scientific works of the seventeenth century. If Gradić's treatise is to be contributed to the historical development of the problem of the infinite, then, first of all, it should be examined in relation to Galileo's and Newton's understanding of infinite quantities and infinite processes.

In my study of Gradić's understanding of the infinite, I shall focus on the analysis of the published version of the treatise *De loco Galilaei*, drawing

* The author is greatly indebted to the staff of the Biblioteca Apostolica Vaticana, the Library of the Pontificia UniversitB Gregoriana in Rome, and the Library of the Friars Minor (Knjižnica Male braće) in Dubrovnik for their generous and hospitable assistance. The rare editions of works from the seventeenth century quoted and explored in this paper are kept in these three libraries: Bonaventura Cavalieri, *Geometria indivisibilibus continuorum, nova quadam promota* (Bononiae: Ex Typographia de Ducijs, 1653), sign. BAV Racc. Gen. Scienze IV-465; Honoratus Fabri, *Tractatus physicus de motu locali, in quo effectus omnes, qui ad impetum, motum naturalem, violentum, et mixtum pertinent, explicantur, et ex principiis Physicis demonstrantur* (Lugduni: Apud Ioannem Champion in fore Cambij, 1646), sign. BPUG Mag 620 L 36; Stephanus Gradius, *Dissertationes physico-mathematicae quatuor* (Amstelodami: Apud Danielem Elsevirium, 1680), sign. KMB 34-II-16.

[1] Stephanus Gradius, »Dissertatio III. De loco Galilaei, quo punctum lineae aequale pronuntiat,« in Gradius, *Dissertationes physico-mathematicae quatuor* (Amstelodami: Apud Danielem Elsevirium, 1680), pp. 39-54.

[2] Cfr. Mirko Dražen Grmek, »L'apport de Dubrovnik aux sciences mathématiques et physique jusqu'B l'époque de Bošković,« in *Actes du Symposium international R. J. Bošković 1961* (Beograd: Conseil des Academies RPFY, 1962), pp. 243-254, on p. 252; Žarko Dadić, »Položaj matematike, fizike i astronomije u kulturnoj prošlosti Dubrovnika i doprinos Dubrovčana tim znanostima (do početka 19. stoljeća),« *Rasprave i građa za povijest nauka* 3 (1969), pp. 5-73, on p. 49; Zdravko Faj, *Kritičko istraživanje doprinosa Dubrovčanina Stjepana Gradića razvoju matematike i fizike u 17. stoljeću*, Ph.D. thesis (Zagreb: Prirodoslovno-matematički fakultet, 1978), particularly the chapter »O jednom paradoksu kojim Galilei dokazuje jednakost točke i crte« on pp. 108-117; Žarko Dadić, »Znanstveno djelovanje Stjepana Gradića u Rimu,« in *Povijest egzaktnih znanosti u Hrvata*, Vol. 1 (Zagreb: Liber, 1982), pp. 216-231, on p. 218; Ivica Martinović, »Galilejev paradoks o jednakosti točke i crte u prosuđivanjima Stjepana Gradića i Ruđera Boškovića,« in Žarko Dadić (ed.), *Zbornik radova o dubrovačkom učenjaku Stjepanu Gradiću (1613-1683) u povodu 300. obljetnice smrti* (Zagreb: Hrvatsko prirodoslovno društvo, 1985), pp. 49-70; Stjepan Krasić, *Stjepan Gradić (1613-1683): Život i djelo* (Zagreb: JAZU, 1987), pp. 496-498; Ivica Martinović, »Cavalieri, Fabri i Gradić o Galileievu paradoksu posude,« *Anali Zavoda za povijesne znanosti HAZU u Dubrovniku* 30 (1992), pp. 79-91.

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STEPHANI GRADII

Bibliothecæ Vaticanæ Præfecti

DISSERTATIO III.

De loco Galilæi, quo punctum lineæ æquale pronuntiat.

Quanquam ſcripta Galilæi noſtri, Octavi doctiſſime, tum cæteris admirabilis eloquentiæ laudibus, tum ea, quæ in explicandi facilitate, illuſtrique & perſpicuo dicendi genere ſita eſt, maxime præſtant; cum tamen varia, & latiſſime patens ejus Viri doctrina ſit; non oſcitantem lectorem imprimis autem ſcitum, prudentemque deſiderant, præcipue vero in ea parte, qua viros doctos inter ſe colloquentes inducit, nec ita facili negotio judicandum permittit quamnam ex variis

C 3 illic

Figure 1. The first page of Gradić's only mathematical treatise. Stephanus Gradius, *Dissertationes physico-mathematicae quatuor* (Amsterdam, 1680), p. 39.

special attention to the historical development of the problem of the infinite in regard to the logical structure and methodological options of Gradić's geometrical discourse. My intention is to investigate the genesis of Gradić's treatise as much as its latest and published version will enable me. Finally, I shall point to the mathematicians who could have influenced Gradić to study the paradox of the bowl. Comparative research of the same problem reveals two of his contemporaries: Cavalieri and Fabri.

Therefore, this paper includes:

(1) a presentation of Galileo's paradox on the equality of point and line;
(2) a description of the genesis of Gradić's treatise;
(3) Cavalieri's approach to Galileo's paradox;
(4) Fabri's approach to Galileo's paradox;
(5) Gradić's evaluation of Galileo's paradox; and
(6) an evaluation of Gradić's approach in comparison to the approaches of his contemporaries Cavalieri and Fabri.

1. Galileo's paradox of the bowl: problem and geometric construction

The paradox of the bowl or the paradox on the equality of point and line is included in the first dialogue of Galileo's outstanding work *Discorsi e dimostrazioni matematiche, intorno B due nuove scienze attenenti alla mecanica & i movimenti locali*, incorporated into the vivid exchange of ideas between Salviati and Simplicio on the problem of the composition and resolution of a continuum.[3] Galileo's discussants espouse two conflicting heritages, those of the Neoplatonists and the Peripateticists.

Salviati, a Neoplatonist by scholarly conviction and Galileo's *alter ego*, presents the problem of the resolution of continuum using examples of geometric forms and physical bodies. The first example describes the resolution of the line in two ways. First, the line is divided into countable quantitative parts, i.e. a finite number of parts. By distributing the parts, a greater extension

[3] Galileo Galilei, *Discorsi e dimostrazioni matematiche, intorno B due nuove scienze attenenti alla mecanica & i movimenti locali* (In Leida: Appresso gli Elsevirii, 1638; impression anastaltique Bruxelles: Culture et civilisation, 1966), pp. 25-32; cfr. Galileo Galilei, *Dialogues concerning two new sciences*, translated by Henry Crew & Alfonso de Salvio (New York: Dover, 1954), on pp. 24-31.

cannot be obtained than the one these very parts had prior to the distribution and inserting of the vacua between them. *Segments* must be *recognized* in such parts. And second, a line resolved into parts with no quantity, that is, into an infinite number of indivisibles, is *conceivable*. Thus, a line can be imagined dispersed to infinity by inserting an infinite number of vacua indivisibles. The parts resulting from the partition of the line should be regarded as *points*. However, there is an essential discrepancy between these two partitions of the line: partition into segments is real, whereas partition into indivisibles is imaginary. Salviati approaches the problem of the golden ball in the identical manner.

Simplicio, whom Galileo appointed to represent the Aristotelian tradition, immediately recognizes Democrites' atomistic approach in Salviati's conception of the resolution of the solid into indivisibles. His major obstacle is »the building up of a line out of points, a divisible out of indivisibles, a quantity out of nonquantifiables«.[4] That is a classical Aristotelian objection according to which the part and the whole are not to be described or »defined« by their properties which make them irreducible to one another. The second important aspect of the same problem opens up the question of the number of parts in the partition of a continuum. It seems that Albert of Saxony was the first to present the Peripatetic formula »continuum non dividitur in infinita, sed in infinitum dividitur«. The formula states: by dividing a continuum, one cannot obtain an infinite number of parts in the sense of an actual infinite, but the division of a continuum can continue *in infinitum*, of course, in the sense of a potential infinite. That is the historical background which provokes Simplicio to consider the number of parts as the criterium for the equality of geometric quantities: in what sense can the centre and circumference of the circle be considered equal if the centre is a single point and the circumference comprises an infinite number of points?[5] This is the question with which Simplicio initiates the paradox.

Salviati consents with Simplicio that the understanding of the infinite is an open problem, describing the status of the seventeenth century mathematicians in this manner: »But let us remember that we are dealing with infinities and indivisibles, both of which transcend our finite understanding, the former

[4] Galilei, *Discorsi*, p. 27: »In oltre quel comporre la linea di punti, il divisibile di indivisibili, il quanto di non quanti, mi paiono scogli assai duri da passargli:«

[5] L. c.

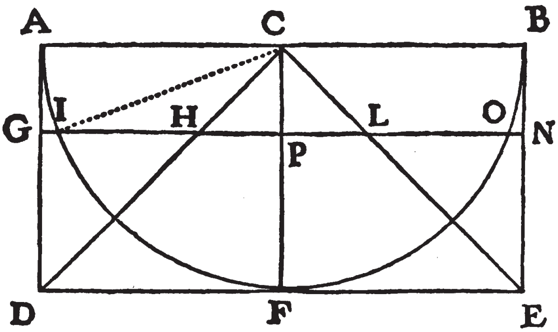

Figure 2. Galileo's paradox of the bowl: A continuous motion of plane GN toward the top of a cone and a crater. Galilei, *Discorsi* (Leiden, 1638), p. 28.

on account of their magnitude, the latter because of their smallness.«[6] There are numerous ways of seeking out the infinite, but Salviati decided to apply »a new marvel«.[7]

In order to elaborate and prove his »new marvel«, Salviati used a diagram (Fig. 2). Rotating about the axis CF, rectangle ADEB, semicircle AFB and triangle CDF form a cylinder, a hemisphere, and a cone. If the hemisphere is extracted from the cylinder, we get a »bowl« (*scodella*). The first step is to prove that the bowl and the cone are equal in volume. Then the bowl and the cone are cut by a plane parallel to the bases of these bodies »at any distance whatever« (*per qualsivoglia distanza*). It should be proved that the plane cuts equal portions CHL of the cone and upper portions GAI,BON of the bowl. In order to prove this, it is sufficient enough to prove the equality of their bases, that is, the circle HL and the circular band GI,ON. The proof was carried out in accordance with the Euclidean method (textual expressions, Pythagoras' theorem, the use of ratio).[8] In modern algebraic notation, it reads:

[6] Galilei, *Discorsi*, p. 27: »mB ricordiamoci, che siamo trB gl'infiniti, e gl'indivisibili, quelli incomprensibili dal nostro intelletto finito per la grandezza, e questi per la lor piccolezza;« Cfr. Galilei, *Dialogues concerning two new sciences*, p. 26.

[7] Galilei, *Discorsi*, p. 27: »... alcuna mia fantasticheria se non concludente necessariamente, almeno per la novitB apportatrice di qualche maraviglia:«

[8] Galilei, *Discorsi*, on pp. 30-31.

$$IC^2 = IP^2 + PC^2$$

$$AC^2 = IP^2 + PH^2$$

$$GP^2 = IP^2 + PH^2 / \$ \ 4$$

$$GN^2 = IO^2 + HL^2$$

$$GN^2 - IO^2 = HL^2$$

The following step leads directly to a paradoxical consequence. A cutting plane of the bowl and the cone reaches the top of these bodies succesively (*succesivamente*). In the course of the process, the sections of the bowl and the cone are always equal, so the portions of cone and the bowl are equal in volume. Therefore, the ultimate elements of the process are to be equal, too. On the contrary, the end of the process results in the point C being the sole residue of the cone and in the circumference of the circle, marked AB in the diagram, being the sole residue of the crater. Upon this proof, its propositions and consequences, Salviati developed a new approach to understanding equality in geometry which would include the paradoxical equality of point and line.[9]

Following the presentation of the paradox, the discussion between Salviati and Simplicio brings up the problem of the composition of continuum as a problem which inspired Salviati's geometric construction with the paradoxical consequence. According to Salviati, the divisible quantity is composed of indivisibles, moreover, of an infinite number of indivisibles (*ma si bene infiniti*), and the infinite herein is the actual infinite. In accordance with this Neoplatonic view, the infinite and the indivisible are accepted as actually existent: the infinite (*l'infinito*) as an actual infinitely great value expressed by number and the indivisible (*l'indivisibile*) as an actual infinitely small value expressed by quantity. On the contrary, Simplicio expresses doubt as to whether the infinite has been understood correctly. What happens when we establish that one line is greater than the other? If every line consists of an infinite number of points, does the infinity of the points of the greater line exceed the infinity of the smaller one? Do we get an infinite greater than infinity itself? Along with his already stated arguments against the indivisible, Simplicio also adds the problem of transfinity.

9 Galilei, *Discorsi*, p. 29: »Li quali perche non si devon chiamare eguali, se sono le ultime reliquie, e vestigie lasciate da grandezze eguali?«

Therefore, the paradox does not contribute to the understanding between the discussants in Galileo's dialogue. Moreover, the two opposed standpoints, Neoplatonic and Peripatetic, represent an inviting challenge for further mathematical research of continuity and infinity. Although Galileo's paradox caused no acute crisis in mathematics, this problem has remained quite receptive among the mathematicians over the years. To my knowledge, the proof applied for determination of the volume of solid, leading to Galileo's paradox on the equality of point and line, has so far been studied by Luca Valerio, Galileo Galilei, Bonaventura Cavalieri, Evangelista Torricelli, Marin Mersenne, HonorP Fabri, Stjepan Gradić, Ruđer Bošković and Bernhard Bolzano.

2. The genesis of Gradić's treatise

The opening and closing passages of the geometrical treatise *De loco Galilaei* confirm Gradić's complete knowledge of mathematical studies after Galileo. During Gradić's intensive research in exact sciences (1656-1660), Galileo's ideas came to life through the scientific work of his pupils, primarily in the circle of the famous Florentine *Academia del Cimento*. This was a unique academy of natural sciences which, in its short history (1657-1666) and under the motto *provando e riprovando*, encouraged an experimental approach in applied mathematics, physics, astronomy, and physiology. Scholars with whom Gradić exchanged views on mathematical and physical problems and gathered in that circle were: Vincenzo Viviani, Giovanni Alfonso Borelli, Michelangelo Ricci, HonorP Fabri, Ottavio Falconieri, and Lorenzo Magalotti. A number of them took active part in the literary-scientific circle gathered around Queen Cristina of Sweden, with Gradić included. Galileo's authority was unquestionable within the scientific circles Gradić was acquainted with, and Galileo's solutions served as a criterion of exactness. Gradić phrased it in the following way: »One should boldly say that what is not exact in the balance of perfect reason, can, by no means, be of Galileo.«[10]

In spite of Galileo's undeniable authority, Gradić realized that parts of Galileo's work offer but mere hints of the problems yet to be solved, and therefore

[10] Gradius, »De loco Galilaei,« p. 54: »ita si quid ad trutinam perfectae rationis exactum non esset, Galilaei nullo modo esse posse audacter dicendum sit.«

require »a wise and experienced reader«, »provoking scholars to dispute«.[11] Few of the learned men dared to banter Galileo or reproach him for evident paralogism. Gradić admits having discussed the problem of Galileo's paradox on the equality of the point and the circumference of a circle with the scholars mentioned.[12] Gradić does not name them, but they ought to be looked up among the Roman scholars from Queen Cristina's circle.

The one name that Gradić does point out is that of *Octavius*. He is the addressee of Gradić's treatise. He addresses him on several ocassions: »Octavi doctissime«, »mi Octavi«.[13] This excessively learned *Octavius*, close to Gradić, obviously of Florentine origin, for he had been invited to the magnificent party in honor of the Duke's engagement in Florence, was the obscure Ottavio Falconieri, whom reference books describe as a great expert in Greek and Roman antiquity but not as a mathematician.[14] Gradić's addresses to Falconieri are of the utmost value, for they reveal a series of details concerning the genesis of the treatise *De loco Galilaei*.

Where and when was the treatise *De loco Galilaei* written? Gradić first decided to examine »the most difficult place« in Galileo's *Discorsi*, »the masterly dispute on the nature of the infinite« which »induces shrewd and profound discussion on the difference between the finite and the infinite«.[15] This was the subject of Gradić's thorough discussion with Ottavio Falconieri and Giulio Montevecchio in the studio of the former (*in tuo museo*),[16] and later, with other anonymous scientists expressing critical attitudes towards Galileo's text.[17] Finally, he wrote a treatise in the form of a letter addressed to Falconieri in order to discuss the same subject with »Lorenzo Magalotti, an exquisite and learned young man«.[18] Falconieri was on his way to Florence (*in hoc tuo*

[11] Gradius, »De loco Galilaei,« p. 39.

[12] Gradius, »De loco Galilaei,« p. 53: »Sic ego, mi Octavi, praeter id quod una cum Julio nostro, ut jam dixi, disputavimus super hoc egregii Galilaei pronunciato, non semel postea disputavi cum plerisque aliis qui tanquam erroris, ac manifesti paralogismi convictum Virum doctissimum illoto ore carpere ac vituperare sunt ausi.«

[13] Gradius, »De loco Galilaei,« pp. 39 and 53.

[14] Cfr. *Biografia universale antica e moderna* 19 (Venezia, 1824), pp. 379-381.

[15] Gradius, »De loco Galilaei,« p. 41.

[16] L. c.

[17] Gradius, »De loco Galilaei,« p. 53.

[18] Gradius, »De loco Galilaei,« p. 54: »Haec tu cum Laurentio Magalotto adolescente lectissimo doctissimoque ut communices rogo, & de utriusque sententia ad me aliquid hac de re perscribas;«

Florentino itinere) to take part in the celebration of the Duke's engagement and Gradić expected him to note something on both sentences (*de utriusque sententia*), that is, on the Neoplatonic and Peripatetic sentences of the continuum and the infinite.[19] Gradić, therefore, was awaiting critical opinion on his understanding of infinite quantities and infinite procedures from the very focus of the Florentine scientific milieu, as the supposed Falconieri's fellow-discussant was the very Lorenzo Magalotti (1637-1712), the young industrious secretary of the *Academia del Cimento*, appointed to that post in 1660 upon recommendation of Vincenzo Viviani. Magalotti was Viviani's pupil, deserving special merits for editing the contributions of the Academy to the development of exact sciences in the work *Saggi di naturali esperienze fatte nell'Academia del Cimento* (1666), which paved the way for the European reception of Florentine experiments and interpretations.[20] Besides, Magalotti addressed two letters to Falconieri, later to be published in the collection of his scientific correspondence.[21]

The writing of the treatise itself did not take long, for Gradić marked the beginning of his study on Galileo's paradox as »not long ago« (*nuper*).[22] The end of that period could be drawn by the marriage of princess Marguerita Louise, daughter of the Duke of Orleans, to Duke Cosimo de'Medici of Toscane, later Cosimo III, in Paris on 19 April 1661.[23] Gradić must have completed his text well before the marriage date so that Falconieri could have taken it with him to Florence to await the young married couple. Gradić's treatise *De loco Galilaei* was concluded by April 1661, but was not published in Amsterdam until 1680.

Who were Gradić's contemporaries that might have influenced him to examine Galileo's paradox of the equality of the point and the line? The mathe-

[19] L. c.

[20] *Biografia universale* 33 (Venezia, 1827), pp. 278-280; J. C. Poggendorff, *Biographisch-literarisches Handwörterbuch zur Geschichte der exakten Wissenschaften* 2 (Leipzig, 1863), p. 10; G. Gabrieli, »Accademia del Cimento,« in *Enciclopedia Italiana* 10 (1931), pp. 249-250.

[21] Lorenzo Magalotti, *Lettere scientifiche, ed erudite* (In Venezia: A' spese della Compagnia, 1734), pp. 58-74. On a discussion about the motion of fire and light held at home of Ottavio Falconieri, see p. 26: »Un giorno fra l'altre in Casa del Signor Ottavio, s'entro nel discorso de'movimenti del Fuoco, e della Luce, ...«

[22] Gradius, »De loco Galilaei,« p. 41.

[23] L. A. Muratori, *Annali d'Italia dal principio dell'era volgare* 11 (Milano, 1749), pp. 279 and 284.

maticians who had studied Galileo’s paradox before 1661 and came into contact with Gradić were Bonaventura Cavalieri and HonorP Fabri.

3. *Cavalieri’s approach to Galileo’s paradox*

Having studied the manuscript *Vat. lat. 6917* from Gradić’s legacy to the Biblioteca Apostolica Vaticana, Stjepan Krasić came upon a valuable piece of information, revealing that Bonaventura Cavalieri was Gradić’s professor of mathematics during his schooling at the University of Bologne (1637?-1638).[24] Gradić admitted to have simultaneously studied mathematics and law or at least attended Cavalieri’s lectures in mathematics, which might have aroused his interest in Galileo’s paradox. Apart from this, Cavalieri elaborated his approach to Galileo’s paradox in his already published works and at the earliest in his correspondence with Galileo throughout 1634. Supposing that Gradić, as a student in Bologna in the late 1630s, or as an active scientist in Rome at the beginning of the 1660s, was not familiar with the details of the 1634 correspondence between Galileo and Cavalieri, he could have been directly influenced by Cavalieri’s lectures. Moreover, Cavalieri’s final standpoint was easily accessible from his published works.

The basic proof preceding Galileo’s paradox, Cavalieri laid out in his masterpiece *Geometria indivisibilibus continuorum, nova quadam ratione promota* (1635, 1653), which was first edited immediately before Gradić’s arrival to study courses in Bologna. The third book of Cavalieri’s work, containing »the doctrine on the circle, the ellipse, and the bodies produced from them«[25] by rotation, embodies theorem 5 and the corollary with the stated proof (Fig. 3).[26]

If we apply modern algebraic notation and the symbols in Fig. 3, theorem 5 is expressed as follows:

$$SN^2 = VR^2 - TI^2.$$

24 *Vat. lat 6917*, f. 9r: »... Bonaventurae Cavallerio, magistro olim in Academia Bononiensi meo; ...«; cfr. Stjepan Krasić, *Stjepan Gradić (1613-1683): Život i djelo*, p. 21 and 477.

25 See the title of the third book »Geometriae Cavalerii liber tertius. In quo de Circulo, et Ellipsi, ac Solidis ab eisdem genitis, traditur doctrina,« in Bonaventura Cavalieri, *Geometria indivisibilibus continuorum, nova quadam promota* (Bononiae: Ex Typographia de Ducijs, 1653).

26 Cavalieri, *Geometria indivisibilibus continuorum*, pp. 204-205.

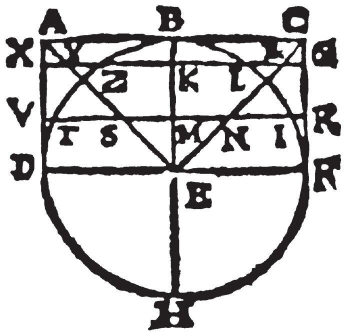

Figure 3. Proof for the equality of the circle SN and the circular band VT, IR. Cavalieri, *Geometria indivisibilibus continuorum* (Bologna, 1653), p. 205.

Cavalieri carries out the proof of the formula with the help of proportion in accordance with elementary Euclidean method, deliberately quoting Euclid's *Elements* in two marginal notes.[27] Those proportions, formulated in sentence sequence and transformed into algebraic notation, read:

(1) $(HE \cdot EB) : (HM \cdot MB) = FE^2 : IM^2$,

(2) $(HE \cdot EB) : ME^2 = BE^2 : ME^2 = RM^2 : (EI^2 - MI^2) = RM^2 : (RM^2 - MI^2)$,

(3) $BE^2 : ME^2 = BC^2 : MN^2 = MR^2 : MN^2$,

(4) $MR^2 : MN^2 = MR^2 : (MR^2 - MI^2)$,

(5) $MN^2 = MR^2 - MI^2 / \cdot 4$

(6) $SN^2 = VR^2 - TI^2$.

It is in this corollary, where Cavalieri introduces figures and the ratios of their areas, that he uses the method of indivisibles, founded in this very work and efficiently applied in planimetry and stereometry. He finds it sufficient to state that the point M was chosen arbitrarily on BE and the base DF serves as *regula* or directrix above which is constructed the rectangle determined by the diagonal AF. He concludes on the equality between the sum of the squares of

27 Cavalieri refers to proposition 5 in the second book and to proposition 4 in the sixth book of Euclid's *Elements*.

all indivisibles forming the trapezium YSNR' and the sum of the squares of all the indivisibles forming the figure constructed as the residuum of the subtraction of the area ZTIL from the paralelogram determined by the diagonal XR. It is stressed that the ratio holds »for any paralelogram«.

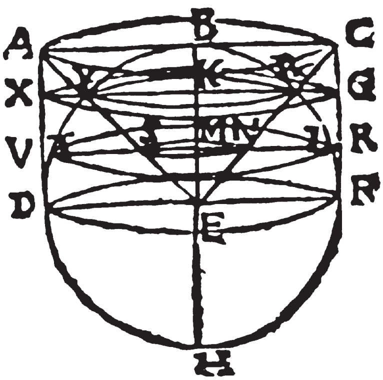

Figure 4. Ratios between the volumes of different solids. Cavalieri, *Geometria indivisibilibus continuorum* (Bologna, 1653), p. 263.

Evidently, the author does not touch the problem of solids yet. That subject is examined toward the end of the third book, displayed in Fig. 4 and proved in a special corollary.[28] In this corollary as well, Cavalieri primarily examines the relation existing between the sums of the squares of all the indivisibles building up areas and in this very case forming the paralelogram AF and the triangle AEC, respectively paralelogram XR and trapezium YSNR'. With the help of this ratio he establishes his statement concerning the ratio of the volumes of the different solids formed by the rotation of the paralelogram AF and the semicircle DBF. That ratio remains constant whenever, due to the rotation of a geometrical form taking part in the initial ratio, a new body, »whatever it may be« (*quodcunque illud sit*), is obtained.[29] Still, Cavalieri does not consider what takes place with those solids at the end of the procedure if their intersections continue to decrease: the triangle SEN on one hand and the plane

[28] Cavalieri, *Geometria indivisibilibus continuorum*, p. 263.

[29] L. c.

figure resulting from the subtraction of the area DTIF from the paralelogram VF, and forming »a bowl« by rotation on the other hand. The reason why Cavalieri acts contrary to Galileo's idea becomes evident after one studies his correspondence with Galileo immediately before the first edition of Cavalieri's work *Geometria indivisibilibus continuorum*. The letters addressed to Galileo, dated 2 October and 19 December 1634, reveal that Galileo shared his doubts concerning the paradox of the bowl with his pupil and friend Cavalieri.[30] This means that two famous works, Cavalieri's *Geometria* (1635) and Galileo's *Discorsi* (1638), embody the very standpoints of the two scientists that chrystalized in and after their correspondence.

Cavalieri's first objection, expressed in his letter of 2 October 1634, focuses on the dimensions of geometrical entities at the end of the procedure. Galileo informed him in writing of the proof of equality of two solids in volume: the cone formed by the rotation of the triangle AEC and the »bowl« formed by the rotation of a figure obtained when the semicircle DBF is substracted from the paralelogram AF (Fig. 4). Galileo repeated the same proof in the case of solids formed as the result of a cone and a bowl being cut by a plane at level XG or even closer to the apex of the cone at level VR. In order to prove the equality of the two solids in volume, he proved the equality of the areas of the bases of those solids. Finally, Galileo concluded, asserting the equality between the apex of the cone, point E, and the upper edge of the bowl, the circumference of the circle with the diameter DF. Cavalieri disputed this statement, as Galileo concluded that the equality of surfaces implies the equality of geometrical entities with smaller dimensions.[31]

Cavalieri's second objection, described in the letter of 16 December 1634, originates in his understanding of the method of indivisibles, and is therefore more far-reaching. According to the method of indivisibles, or as Cavalieri put it, »secondo le mie definitioni«, the idea of all the lines of a surface or all the surfaces of one solid does not include the geometrical entities at the limits of the figure, although they may seem to be of the same kind. The explanation can assure a mathematician and a physicist alike, as Cavalieri adjusted it to

[30] Cfr. Giulio Giorello, »Galilee, Cavalieri et les indivisibles,« *Actes 'Jourees Galilee'*, Cahier 9 (1980), pp. 1-21, on pp. 16-17; François de Gandt, »Les indivisibles de Torricelli,« *Cahiers du Séminaire d'épistémologie et d'histoire des sciences* 17 (1983/1984), pp. 36-118, particularly the chapter »Les indivisibles comme vestiges ultimes et le paradoxe du bol,« on pp. 89-90.

[31] Galileo Galilei, *Opere*, Edizione Nazionale, Vol. 16, pp. 136-137; cfr. Giorello, »Galilee, Cavalieri et les indivisibles,« p. 16.

Galileo's scientific interest: »Since I consider all the lines of a plane surface to be the common sections of a plane cutting the figure while moving from one limit to the other or from one tangent to the opposing one, and since the beginning and the end of the motion is not true motion, the extreme tangents cannot be counted among all the lines.«[32]

Cavalieri thus indicates that the question of the limit of a geometrical figure within the method of indivisibles is neither appropriate nor logical, whereas Galileo with his paradox puts forward the very problem. Although the letter containing Galileo's presentation of the paradox to Cavalieri has not been preserved, he must have formulated the problem on the equality of line and point in the same manner as in his work *Discorsi* four years later: »Why shall we not then call them equal, seeing that they are the last traces and remnants of equal magnitudes?«[33]

Unlike Galileo, Cavalieri considered that, within his own mathematical methodology, he could justly conclude that the problem of equal areas was not at the same time the problem of their boundaries. Thus, his own response to Galileo's *dubio della scodella* encourages him in the belief that by creating the method of indivisibles, he reaches a successful method in geometry.

4. Fabri's approach to Galileo's paradox

Towards the end of the treatise *De loco Galilaei,* Gradić stated that during his stay in Rome he met with scholars who shared a critical view on Galileo's paradox of equality of the point and the line and with whom he had disputes on several occasions.[34] One of the scholars who undoubtedly examined Galileo's paradox was HonorP Fabri. He presented his approach to Galileo's paradox in his lectures in physics, *Tractatus physicus de motu locali,* printed in Lyon in 1646.[35] This treatise on local motion was published the same year that he

[32] Galilei, *Opere*, Vol. 16, p. 175; cfr. Giorello, »Galilee, Cavalieri et les indivisibles,« p. 17; de Gandt, »Les indivisibles de Torricelli,« p. 90.

[33] See note 9.

[34] See note 12.

[35] *Tractatus physicus de motu locali, in quo effectus omnes, qui ad impetum, motum naturalem, violentum, et mixtum pertinent, explicantur, et ex principiis Physicis demonstrantur*. Auctore Petro Movsnerio Doctore Medico: cuncta excerpta ex praelectionibus R. P. Honorati Fabry, Societatis Iesu (Lugduni: Apud Ioannem Champion in fore Cambij, 1646).

terminated his lectures in philosophy and mathematics in Lyon and started at the post of penitentiary in the Vatican basilica of St. Peter, where he remained till 1680, producing fruitful research in moral theology.[36] At the time when the treatise *De loco Galilaei* was being written, Fabri and Gradić exchanged scientific correspondence. Fabri's letter to Gradić, dated 7 January 1661, contained Fabri's objections directed toward Gradić's earlier treatise *De causa naturali motus accelerati*.[37] Fabri's objections were most accurately formulated: Whence derives the indifference of solids to motion or stillness? Why does the cause act momentarily? Why is the composition of two motions always possible? Therefore, there is no doubt that Gradić and Fabri, side by side, discussed the open problems of natural sciences. These discussions offered Gradić an opportunity to examine Fabri's understanding of Galileo's paradox and the possible evolution of Fabri's views from the Lyon edition of his *Tractatus physicus* in 1646 to 1661.

In his Lyon lectures, Fabri presents the complete context of Galileo's approach, that is, the whole reasoning process underlying the conversation between Salviati, Simplicio and Sagredo in Galileo's *Discorsi*. To Salviati's introductory question »how can it be possible to discover an infinite number of vacua within a continuous finite extension?«, Galileo responded using the considerations on the wheel of Aristotle (*rota Aristotelica*) and the paradox of the bowl, which he conceived on that occasion.[38] Fabri was motivated for Aristotle's study on the wheel, according to his own words,[39] thanks to the works of two mathematicians: *Blancanus* and *Mersennus*. The former was, for certain, the Italian Jesuit Giuseppe Blancani (1566-1624), who contributed to the prime research in the history of mathematics with his work *Aristotelis loca mathematica, ex universis ipsius operibus collecta et explicata* (Bologna,

[36] Cfr. Mijo Korade, »Diskusija Stjepana Gradića i HonorⱣa Fabrija o probabilizmu,« in Žarko Dadić (ed.), *Zbornik o dubrovačkom učenjaku Stjepanu Gradiću (1613-1683)*, pp. 99-106, on pp. 101-102.

[37] Cfr. Žarko Dadić, »Stjepan Gradić o problemima mehanike,« in Žarko Dadić (ed.), *Zbornik o dubrovačkom učenjaku Stjepanu Gradiću (1613-1683)*, pp. 35-40, on pp. 36-37.

[38] Galilei, *Discorsi*, pp. 21-32, with the initial question on p. 21: »..., come in una continua estensione finita non repugni il potersi ritrovar' infiniti Vacui:«. This passage is included in a choice of mathematical sources: Galilei, »On infinities and infinitesimals,« in Dirk J. Struik (ed.), *A source book in mathematics, 1200-1800* (Princeton: Princeton University Press, 1986), pp. 198-207.

[39] Fabri, »Digressio de rota Aristotelica,« in Fabri, *Tractatus physicus de motu locali*, pp. 339-346, on p. 339: »Aristoteles hanc difficultatem habet, quaest. 24. Mechanicorum, quam etiam explicat Blancanus, proponitque Mersennus in praefatione suae versionis mechanicarum Galilaei;"

1615),[40] and the latter was a French Minorite, Marin Mersenne, who wrote about the wheel of Aristotle in the introduction to his translation of Galileo's mechanics.

The ninth book of Fabri's work, as suggested by its title, deals with »the motion combined of linear motion and circular motion or of several circular motions«.[41] The series of seven theorems at the beginning of the ninth book describes the motion of a wheel rolling over a plane, i.e. a motion composed of linear motion of the centre and the circular motion of the circumference. The digression on the problem to Aristotle's wheel seems quite natural (Fig. 5).[42]

The paradox of the wheel was displayed in question 24 of Aristotle's *Mechanics*. Aristotle's wheel consists of two concentric wheels with centre A and radii AB and AC where AC = 2 AB, which roll for the length CE equal to the arc CH, the quarter of the circumference. After this rolling, radius AH coincides with the line GE and the point D on radius AH reaches the position of the point F on radius GE. From the standpoint of Aristotle's *Mechanics* it is evident that single points of arc CH correspond to single points on the seg-

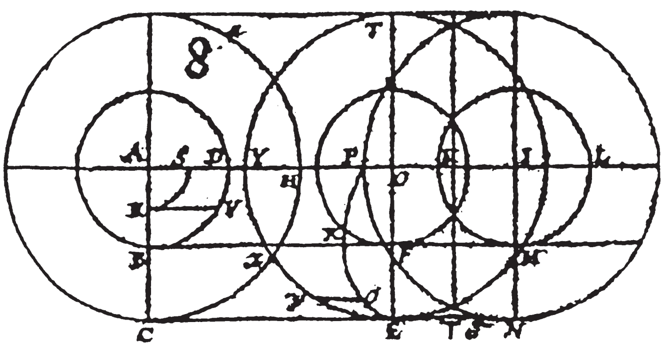

Figure 5. The wheel of Aristotle. Fabri, *Tractatus physicus de motu locali* (Lyon, 1646), Tab. 4, Fig. 8.

40 Cfr. Gino Loria, *Storia delle matematiche dall'alba della civiltB al tramonto del secolo XIX* (Milano: Cisalpino-Goliardica, 1982), p. 368.

41 Fabri, »Liber nonus, de motu mixto ex recto, et circulari, vel ex pluribus circularibus,« in Fabri, *Tractatus physicus de motu locali*, pp. 344 sqq.

42 Fabri, »Digressio de rota Aristotelica,« p. 339; see note 39.

ment CE, but it is doubtful whether certain points of the arc BD of the smaller circle correspond to singular points of the segment BF. In Fabri's words:

»Moreover, the source of difficulty is that BF is twice the arc BD, so that singular points of the arc BD correspond in descent to singular points of BF, or each singular point on BD corresponds to two points on BF, or alternate points on BF remain absolutely intact by skipping. But it seems that none of this can be said.«[43]

From the mathematical point of view, Fabri's elaboration of the reason why a one-to-one correspondence cannot be established between the points of the arc BD and the points of the segment BF remains questionable. If the arc BD comprises as many points as on the segment BF which is equal to arc CH, it would imply that the arc that is half as long is equal to one twice its length. Fabri thus applied the reduction of a one-to-one correspondence to the relation between the part and the whole.

In Galileo's *Discorsi*, a discussion of Aristotle's wheel preceded his presentation of the paradox of the bowl, and Fabri followed the same order.[44] In the presentation of Galileo's paradox of the bowl, Fabri firstly criticized Galileo for inheriting Valerio's proof of the equality of cone and crater only to be followed by his own argumentation for the equality of these two volumes (Fig. 6). Fabri's proof of the equality of the volumes is based on the method of indivisibles: the complete cone consisting of all the circles is equal to the complete round razor which consists of all the circular bands. Fabri does not take into consideration the key issue of *how* the cone can be built up of all the circles resulting as sections of the plane parallel to the base of the cone. When he claims that the section of the plane is always a surface and never a point, Fabri fails to elaborate the method.

Both paradoxes, Aristotle's wheel and Galileo's paradox of the bowl, help Fabri toward the same purpose as they did Galileo. Fabri wishes to elaborate his own view on the problem of *compositio et resolutio continui* and therefore he focuses on the problem illustrated by Aristotle's wheel. From the math-

[43] Fabri, »Digressio de rota Aristotelica,« p. 340, n. 3: »Porrb caput difficultatis potissimum in eo positum est, quod BF sit dupla arcus BD; igitur vel singula puncta arcus BD respondent in decursu singulis BF, vel singula BD respondent duobus BF, vel alterna puncta BF saltuatim remanent penitus intacta; sed nihil horum dici posse videtur:«.

[44] Ibid., pp. 341-342, nn. 8-10.

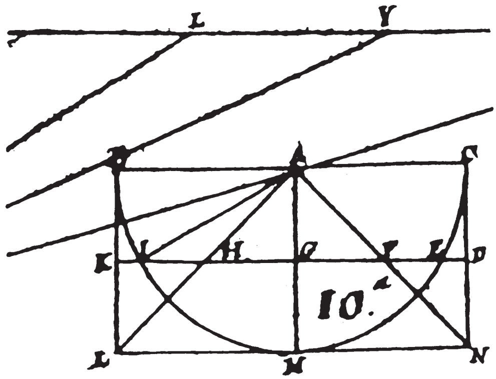

Figure 6. The equality of point and line. Fabri, *Tractatus physicus de motu locali* (Lyon, 1646), Tab. 4, Fig. 10.

ematical standpoint, this problem can be formulated as the question of the existence of correspondence between the points of two unequal segments and from the physical point of view, it is related to the nature of circular motion. In order to solve the controversy, Fabri evaluates several hypotheses. According to the first one, a continuum is composed of mathematical points; according to the second, of proportional parts which are actually infinite in number; and according to the third, of physical points or parts which are potentially divisible to infinity.[45] The first hypothesis, regardless of whether it is considered that a finite or infinite number of mathematical points exists, does not offer an exact answer to the correspondence between the points of the arc BD and the twice greater segment BF. The second hypothesis introduces a somewhat dubious idea of the contact at an undetermined part (*contactus in parte indeterminata*). The remaining third hypothesis, relating to the physical points, is not void of criticism either. The major objection concentrates on how to achieve correspondence between the curved physical point of the arc BD and the plane physical point of the segment BF, when it is clear that the curved quantity cannot equal or commeasure with the plane quantity. At this point, as well as in the consideration of the second hypothesis, Fabri directs toward a more thorough study of these questions in his metaphysics. The final solution

[45] Fabri, »Digressio de rota Aristotelica,« p. 342, n. 11: »iuxta hypothesim punctorum mathematicorum«; p. 342, n. 12: »iuxta hypothesim partium proportionalium infinitarum actu«; p. 342, n. 14: »iuxta hypothesim punctorum physicorum, vel partium divisibilium in infinitum potenti>«.

of the controversy of composition and resolution of the continuum, Fabri preferably seeks in metaphysical arguments rather than in mathematical proof.

5. Gradić's approach to Galileo's paradox

The relation between Galileo's authority and Galilean criticism in the Roman scientific milieu as well as the circumstances which influenced Gradić's research into Galileo's paradox are the prime topics of the introduction and closing of the treatise *De loco Galilaei*. The principal part of the discourse embodies Gradić's approach to ideas and methods he had come across in Galileo's analysis of the infinite. These arguments relate to: (1) the meaning of the final consequence of the paradox, (2) the concept of the uniform procession and (3) the method for measuring volume with a curved limit. The presentation and evaluation of the former will follow in the stated order.

5.1. The meaning of the final consequence of the paradox

Gradić's first objection focuses on the choice of dialogue as a convenient literary genre for disputes on open physico-mathematical problems. Gradić is aware of Galileo's taking the liberty of free conversation in order to display his views, still unwilling to defend the credibility of his paradox in the sense *pro aris et focis*.[46] The dialogue form enables the presentation of different opinions leading to diverse conclusions. This form reduces the speakers' seriousness in philosophical and mathematical deduction. The form of dialogue welcomes witty remarks and surprise effects. According to Gradić, friendly conversation among scholars approaches the nature of poetry.[47]

Gradić commences the elaboration of Galileo's paradox by declaring that the statement on the equality of point and line is far from true.[48] Its untruthfulness is even more evident after a close study of its consequences. If it were so

[46] Gradius, »De loco Galilaei,« p. 42.

[47] Gradius, »De loco Galilaei,« p. 52: »qua in re hoc scribendi genus non multum discrepat B nebre poëtarum, cum alioquin etiam in eo quod familiaria doctorum hominum colloquia imitanda suscipit, proxime ad poëtices naturam accedit, ...«

[48] Gradius, »De loco Galilaei,« p. 42: »Id enim primo B veritate longissime distat, ...«

that the point is equal to the line, then, in the same way, the line is equal to the surface, the surface is equal to the solid. The statement on the equality of point and line consequently means that the unmeasurable point is equal to the tridimensional geometric quantity, therefore, it contradicts Euclid's definition of point.[49] With the intention of reaching for the ultimate consequence of Galileo's paradox, Gradić simultaneously interprets the very definition of point by emphasizing the contradiction in the statement on the equality of point and solid using the concept of measure or dimension.[50] Gradić's expression for point is an object with no measure (*res nullius mensurae*) and for the solid, a quantity of three dimensions (*quantitas trina dimensa*).

Herein Gradić abandons the common Latin translation of Euclid's definition of the point, which bears essentially the concept of the part: »Punctum est, cuius pars nulla est.«, and as a masterly expert in Greek and geometry, he understands the term *to meros* as measure or dimension. Although this comprehension does not include modern mathematical contents of these concepts, it is an unavoidable fact that Gradić, in the midst of the popularity of the method of indivisibles, finds it more accurate to state that the point has no dimension nor measure rather than being indivisible. In other words, Gradić prefers to state that the point is not defined by the process of measuring, but claims the same for the process of dividing. At the very core, this means the dispute of the indivisible.

5.2. The character of uniform procession

According to Gradić's comprehension, the key procedure in Galileo's proof of the equality of point and line is *uniformis processio* as a procedure carried out for the bases of cone and bowl. The original Latin term indicating this process has two basic meanings:

(1) uniform progression, uniform passage;
(2) uniform disappearance, evanescence.

[49] Cfr. the chapter »Euclid's *Elements*,« in John Fauvel and Jeremy Gray (eds), *The history of mathematics: A reader* (London: Macmillan Press, 1988), pp. 99-147, on p. 100: »A point is that which has no part.«

[50] Gradius, »De loco Galilaei,« p. 42: »ac proinde punctum ipsum, hoc est rem nullius mensurae quantitati trinae dimensae statuemus aequalem:«

In my opinion, by choosing the new term, Gradić successfully embodies both aspects of the process described:

(1) continuous increase of the considered elements or their progress in the sense of number of elements if examined in their entirety;

(2) uniform disappearance of elements when an element is observed separately.

Therefore, in further text I shall use the term *uniform procession* for the described procedure in order to emphasize and remind the reader of the richness of meaning in Gradić's terminological choice.

The uniform procession of figures forming bases of cone and bowl, Gradić characterizes with the help of geometrical equality in the following way: »If the solids are intersected by any plane GL parallel to the base of the cone, the surface of the circle HI subtracted from the cone is always equal to a responding circular surface GION, named band or ribbon by Galileo.«[51] He examines the equality of surfaces within the uniform procession of geometric creations and finds it justifiable to state that such an equality is »a constant concomitant of the procession« (*aequalitas semper comitans processionem*).[52] Therefore, Gradić does not consider the equality of geometrical figures and solids a mere identity, but dynamically understood equality related with procedure of uniform procession. Nowhere in the treatise does Gradić doubt the proof of the equality of circle and circular band carried out by Galileo using elementary geometrical method, but rather fully concentrates on uniform procession. Evidently, he supposes that the very procedure of uniform procession conceals the reason for the final paradox. Thus he approaches the procedure step by step.

Gradić's first conclusion is trivial, but he composes the idea of uniform procession into the geometrical consequence of equality of the bases: uniform procession does not solely join the equality of sections of cone and bowl carried out by the same plane parallel to the base of the cone, but it is the procedure within which equality of volumes formed by intersections of cone and bowl by the same plane is also established. Gradić states that »the same equal-

[51] Gradius, »De loco Galilaei,« p. 43: »... si secentur quocumque plano GL ad basim coni parallelo, semper superficies circularis HL quae B cono aufertur respondenti sibi superficiei orbiculari GION (quam Galilaeus *nastrum* sive fasciam appellat) semper est aequalis, ...«

[52] Ib., p. 44.

ity« (*eademque aequalitas*)[53] is here repeated, pointing to the character of that geometric equality.

The second conclusion pertains to the limits (*termini*) of equal surfaces, a subject already initiated by Galileo. What can be said about the limits of these surfaces? They are not equal in the course of the procession, so it is logical to expect that they will not be equal at the end of this procedure either. Gradić specifically states that the equality of surfaces does not necessarily result in the equality of limits of equal surfaces, nor the equality of limits of equal volumes.[54] Due to the conciseness of the treatise, he confines himself to the study of the limits of surfaces.

First, Gradić questions the possibility of expressing the relation between surfaces and their limits by dimension, for he rightly perceives that there are no geometric rules by which to administrate the dimensions of the surfaces (*Geometriae praecepta ad dimensiones planorum administrandas*),[55] and according to which it may be concluded from the equality of limits to the equality of quantities limited by these limits. Then he seems to wonder whether uniform procession enables a somewhat less general conclusion, expressing himself in a conditional manner: if the equality of quantities in uniform procession could be deduced to the equality of limits, the very equality by which such procession could be defined, that would still not imply the equality of a point and the circumference of a circle. Gradić’s reason is built on the fact that the limits in Galileo’s geometric construction have a multi-purpose role.[56] The point and the circumference of a circle are not exclusively the limits of the cone and bowl, but they also represent the limits of those surfaces which form the cone and bowl, the limits of the circle and circular band in particular, and they are certainly not equal. The discrepancy in the magnitude (*magnitudinis discrepantia*) is evident by the fact that the limit of the circle is a unique circumference and the limit of the circular band is formed by two circumferences. This discrepancy increases with the progress of procession (*in progressu processionis*) as the limit of the circle decreases with the decrease of the area of the circle, whereas the limit of the circular band increases due to the de-

53 Ib., p. 43.
54 Ib., p. 44.
55 Ib., p. 45.
56 Ib., pp. 45-47.

crease of the area of the band. This suggests that Gradić bears in mind the postulated relationship between geometric equality and uniform procession even when elaborating the inequality of geometric quantities. The inequality of limits is brought to the absurd if observed that the bases of cone and bowl share the circumference DE as a common limit at the beginning of the procession. This statement, however, is only valid if we accept the hypothesis that it is justifiable to neglect point F which is also the limit of the base of the bowl, for the point as a creation without dimension cannot contribute to the magnitude of the limit.

Gradić's conclusions on the limits of geometric creations are directed towards a broader context, inspiring fresh and more general questions: under what conditions is it possible for a geometric property to be deduced from surfaces to solids and likewise from surfaces to lines? What might be concluded about a geometric object whose dimension is greater by one or lesser by one than the dimension of the studied object? These questions have not been directly put, but they result from Gradić's elaboration.

What actually occurs at the end of uniform procession? According to Gradić's understanding, the progress present in uniform procession leads to the *absolute progress*, that is, to the state which he determines as *postquam absoluto progressu*.[57] This state is demonstrated in such a manner that the limit of the circle contracts to the least possible magnitude - the point or, in Gradić's words, to nothing, and the limit of the circular band expands to the greatest possible magnitude. That means that the limits acquire *extreme values* by quantity, and from this standpoint, Gradić's statement of the limits acquiring *equal effects* could then be accepted.

What mathematical sense lies behind this phenomenological description of the absolute progress? The answer should be sought in comparison with a similar expression in Gradić's treatise *De causa naturali motus accelerati*.[58] The *absolute progress* from the treatise *De loco Galilaei* is the term for the result of the approach to the point in the presentation of Galileo's paradox of

[57] Ib., p. 47.

[58] Stephanus Gradius, »Dissertatio II. De causa naturali motus accelerati & aequalibus ejus in descensu corporum gravium ad aequalia momenta temporum incrementis.,« in Gradius, *Dissertationes physico-mathematicae quatuor* (Amstelodami: Apud Danielem Elsevirium, 1680), pp. 22-38.

the bowl, just as the *absolute,* and at the same time the perfect, *triangle* from the treatise *De causa naturali motus accelerati* describes the result of the limiting process in the proof where the path crossed in a uniformly accelerated motion equals the area of Galileo's triangle in its numerical value.[59] Both cases pertain to the contraction of the line to the point as a form of realization of the limiting process. In the treatise *De loco Galilaei*, the contraction of the circumference to the point observed in the projection seems like the contraction of segment HL to the point C. In the treatise *De causa naturali motus accelerati*, Galileo's triangle is examined first, i.e. the right triangle of which one cathetus represents the interval of time, and the other represents the velocity acquired at the end of the interval. Then all lines drawn from equidistant points on the cathetus representing the time interval determine a denticulate figure, closer to the triangle as the time intervals shorten. The limiting process is finally formulated: »Supposing that such a division is divided into an infinite number of intervals, and consequently, presuming that the initial speed is lower than all the possible ones and contracted to the point which expresses the prime instant of motion, that is, the rest itself from which the given motion initiates,...«[60] On the basis of present knowledge it must not be claimed, but is justifiable to propose, that the contraction of the segment to the point was a possible source of Gradić's interest for Galileo's paradox of the bowl, the desire which urged him to a more profound understanding of the limiting process. By exploiting the term *absolute*, it is certain that in both cases Gradić characterizes the process of contraction to a point in the same manner: as the approach to a limit. This was truly a significant achievement for the year 1661, as it was conceived after Galileo, who made no attempt to determine mathematically the limiting process due to the actual understanding of the infinite. It also appeared before Newton, who was the first to formulate the approach to

[59] Gradius, »De causa naturali motus accelerati,« pp. 35-37. Cfr. the evaluations of this Gradić's treatise in: Faj, »Galileieva i Gradićeva interpretacija jednoliko ubrzanog gibanja i tzv. Galileievog trokuta,« in Faj, *Kritičko istraživanje doprinosa Dubrovčanina Stjepana Gradića razvoju matematike i fizike u 17. stoljeću*, pp. 88-108; Dadić, *Povijest egzaktnih znanosti u Hrvata*, Vol. 1, pp. 228-229; Krasić, *Stjepan Gradić (1613-1683): Život i djelo*, pp. 490-494; Zdravko Faj, »Stav Stjepana Gradića o primjeni nedjeljivih dijelova (indivizibila) u fizici,« *Anali Zavoda za povijesne znanosti HAZU u Dubrovniku* 31 (1993), pp. 31-44, on pp. 36-40.

[60] Gradius, »De causa naturali motus accelerati,« p. 35: »..., adeoque posita tali sectione infinitis intervallis distincta & consequenter posita illa prima velocitate omnium possibilium minima, & in punctum redacta, per quod exprimitur primum instans motus, hoc est quies ipsa, B qua datus motus incipit, ...«

a limit by means of a mathematical method called the method of the first and last ratios.[61]

5.3. *The Archimedean method for the determination of the volume of a solid with a curved limit*

With the explanation of the paradox, Galileo stated that a different approach in determining the equality in volume of the cone and the bowl was possible, and that is the proof which Luca Valerio employed in the treatise *De centro gravitatis solidorum* (1604). The method of measuring surfaces and solids with curved limits employed by Valerio is based upon atomistic ideas of geometric space and can be traced to Democrites' problem: »If a cone is intersected by planes parallel to its base, how are we to consider the areas of section: equal or unequal?«[62] According to the preserved fragment, Democrites immediately spotted that the sum of all sections in the case of unequal surfaces produces an irregular solid with numerous indentities, whereas in the case of equal surfaces it produces a cylinder. The latter case is obviously senseless. He was the first to question the role of the denticulated figure for measuring the solid limited by a curved surface. These initial propositions were implanted into Archimedes' fruitful mechanical, or rather static, studies in which he solved the problem of the quadrature of curved surfaces, the quadrature of the parabola in particular, as well as the problem of the cubature of corresponding solids lacking geometric rigor, which he reserved solely for the method of exhaustion. The West was introduced to Archimedes' principal works as late as the second half of the sixteenth century through translations and comments. Luca Valerio played an outstanding part in the revival and modernization of Archimedes' work, and Galileo rightfully calls him »nuovo Archimede dell'etB nostra«.[63] In his famous work *De centro gravitatis solidorum*, Valerio

[61] Cfr. Isaac Newton, »De methodo rationum primarum et ultimarum, cujus ope sequentia demonstrantur,« in Newton, *Opera quae exstant omnia II. Philosophiae naturalis principia mathematica*, l. 1, sect. 1, Faksimile-Neudruck der Ausgabe von Samuel Horsley, London 1779-1785 (Stuttgart-Bad Cannstatt: Friedrich Fromann Verlag, 1964), pp. 30-41.

[62] Cfr. Oskar Becker, *Grundlagen der Mathematik in geschichtlicher Entwicklung* (Frankfurt: Suhrkamp, 1975), p. 56.

[63] Galilei, *Discorsi*, p. 30.

performed systematic research into all spheroids and conoids, their sections resulting from the intersection of the plane perpendicular to the axis, all founded on Archimedes' example of a parabolical conoid. Valerio applied circumscribed and inscribed denticulate figures composed of a great many equally wide rectangles.

In his presentation of the paradox, Galileo explicitly cites Valerio's proof of equality of the cone and the bowl, which includes the apprehension of integration in a rudimentary atomistic sense. Following Galileo, Gradić also mentions the same proof,[64] but unlike Galileo, he produces an elaborate study in search for the paradox. Here is the basic idea of Valerio's proof interpreted by Gradić:

»That most serious author [= Luca Valerio in his treatise *De centro gravitatis solidorum*] there proves the equality between the crater and the cone by dividing the cone into *many* cylinders by means of equidistant planes parallel to the base of the cylinder, and the crater into *just as many* cylindrical orbs (by cylindrical orb I refer to such a solid, which remains of the greater cylinder after the subtraction of the lesser cylinder having the same axis), so that *every* single cylinder, a component part of the cone, corresponds to a cylindrical orb of the same magnitude. The equality of *singular* parts results in the equality of *all* the parts and thus the equality of cylinders and cylindrical orbs follows the equality of the crater and the cone *just as* it is amongst the elements. Even if by the above-mentioned sections the established cylinders are arranged on one side and the cylindrical orbs are arranged on the other, neither do *all* the cones build up the *whole* cone, nor do *all* the cylindrical orbs build up the *whole* bowl, for the attached cylinders or cylindrical orbs of this kind do not coalesce, in the case of the cone, into *quite* a conical surface, or in the case of crater, so to speak, into a *perfectly* craterial surface. By interposing the established pieces onto both sides, the surfaces become *denticulated in a certain manner*. The same would occur if these pieces were not intervened at all, seeing that such an equality of singular cylinders and orbs always proceeds. If such multiplication of sections were to increase to *any number whatsoever*, the following succession would be unavoidable: such a decrease of singular pieces that all of them put together at the same time build a quantity minor to

[64] See the same proof in: Luca Valerio, *De centro gravitatis solidorum* (Romae, 1604), l. 2, prop. 22; Galilei, *Discorsi*, p. 30; Gradius, »De loco Galilaei,« pp. 47-48.

any given quantity. For this reason, never can it occur that the denticulate figure composed of *however many* cylindrical orbs, approaching *any approximation* of the crater, is not equal to the other equally denticulate figure composed of *just as many* cylinders, the figure which *with equal approximation* approaches equality with the cone.«[65]

The first part of the proof takes into consideration the division of the cone and the crater sharing a common base into a finite number of parts (Fig. 7). The cone and the crater are intersected by equidistant planes, parallel to the common base, the former into cylinders and the latter into cylindrical orbs. The terms expressing quantity, such as *many* cylinders and *just as many* cylindrical orbs, mean that the intersection results in a *finite number* of parts. The *equal number* of parts is expressed by an additional concept of one-to-one correspondence between cylinders and cylindrical orbs. These parts have equal bases and equal heights, provided by the construction, and therefore have equal volumes. In accordance with Euclid's second axiom on the equality of quantities, the sums of a finite number of equal volumes are mutually equal. But such volumes are not at the same time volumes of the crater and the cone, but are denticulate volumes; their planes are not complete and perfect, and this statement is taken for granted if we know how and when, according to Gradić, the surfaces become complete and perfect: after the completion of the procedure of approaching the limit.

How does this transition from a finite to an infinite number of parts of cone and crater take place?

(1) The number of sections increases to any number, that is, has a tendency of becoming infinitely great. Thus, »any number of cylindrical orbs« are formed, and »just as many cylinders« that are quantitative expressions also denoting progress into infinity in the sense of the potential infinite. However, the equality of the denticulated volumes is not in question; what is more, the equality of the volumes is preserved at the limit, too.

(2) The cylinders and cylindrical orbs, component parts of the cone and the crater examined separately, decrease in proportion with the increase of sections, therefore, become arbitrarily small.

[65] Gradius, »De loco Galilaei,« pp. 48-50. I used italics to point out all the concepts essential for the mathematical understanding of this passage. See Appendix at the end of this article.

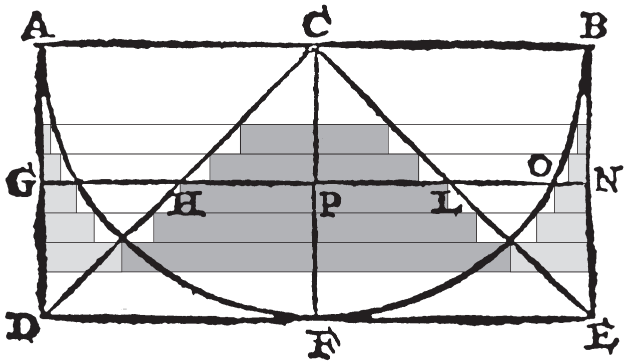

Figure 7. Valerio's proof of the equality of the volumes of the cone and the crater in Gradić's interpretation: measurement of volume by means of inscribed denticulated solids. According to Stephanus Gradius, »De loco Galilaei, quo punctum lineae aequale pronuntiat,« in Gradius, *Dissertationes physico-mathematicae quatuor* (Amsterdam, 1680), figure on p. 43.

(3) The sum of an infinite number of sections is smaller than any given quantity. In other words: if the number of parts increases to infinity, the sum of those and such parts is not an infinite value, but the very opposite. The sum comprises all the parts, infinitely many taken simultaneously, that is, the sum of the infinite series is actually understood.

(4) The difference between the volume of such a denticulated solid formed in the aforementioned manner and the volume of any of the initially given solids, crater or cone alike, becomes *whichever*, arbitrarily small. This difference is understood as an actual infinitely small quantity as the indentedness of the volume, in Gradić's interpretation of Valerio's proof, does not cease to exist on the limit either. Moreover, the equality of the volumes is expressed in another appropriate manner. Approximations (*proximitates*), simultaneously studied in relation to the cone and the crater, in the course of the process remain constantly equal.

After the presentation of Valerio's proof, Gradić approaches its evaluation. Like Salviati in Galileo's first dialogue, Gradić also repeatedly stresses that in the course of the process of approaching the limit, the considered geometric equality preserved. The reason is evident: the equality during the limiting process would imply the equality on the limits, the equality of the point and the circumference of the circle. However, Gradić disputes Salviati's understanding of the equality stated in the question: why not name these limits equal?[66] The existing difference between the limits of the cone and crater should be accepted as obvious, and Gradić therefore searches into the *relation between the limit and the object it limits* as a possible source of Galileo's paradox. He sets two mutually opposing propositions using Euclid's axioms on the equality and inequality of quantities:

(A2) If equals be added to equals, the wholes are equal.

(A3) If equals be subtracted from equals, the remainders are equal.

(A4) If equals be added to unequals, the wholes are unequal.[67]

The application of these axioms implies that Gradić understands Euclid's concept of the equality of quantities as the only valid concept of the equality of geometrical quantities.

The first supposition: the point and the circumference of the circle *are* respective component parts of the crater and the cone.[68] They should be added to the sums of the infinite number of mutually equal components of the crater and the cone. The construction insures the sums to be equal: it was first stated that the sums of a finite number of components of the crater and the cone are equal, and that the equality of sums persists in the process of increasing the number of terms to any amount. On the other hand, the point and the circumference of the circle are not equal. At the step where the limit of the solid is reached, the point on the cone and the circumference of the circle on the crater, that is, at the step where the cone and crater, according to Gradić, *fully close*, it occurs that two unequal quantities are added to two equal ones and those sums, according to Euclid's fourth axiom on the inequality of quantities, are unequal. Gradić moderates the conclusion: the equality of the cone and the crater is only dubious or there is no stated argument which could be the cause for a definite rejection of the proof of the theorem on the equality of cone and crater.

[66] See note 9.

[67] Cfr. the chapter »Euclid's *Elements*,« in Fauvel and Gray (eds), *The history of mathematics*, pp. 99-147, on p. 102.

[68] Gradius, »De loco Galilaei,« p. 50: »..., vel enim punctum & circumferentia, de quorum aequalitate ille pronunciat, concurrunt tanquam partes ad componendum integraliter craterem & conum, ...«

The second supposition: the point and the circumference of circle *are not* component parts of the cone and the crater.[69] In this case, Gradić considered the sums of the components of the cone and the crater as equal parts, subtracted on both sides from the *whole* of cone and crater. By subtracting parts from the whole, we get remainders (*residui*). By introducing the terms whole, part, and remainder, the considered geometrical problem has been prepared for the application of Euclid's third axiom on the equality of quantities. But Gradić states the very opposite as he foresees a problem in the verification of Euclid's third axiom: »To verify the axiom on the equality of remainders from the equality of subtracted parts from equal wholes, it is necessary for the equal wholes in the comparison to consist of the subtracted parts and the remainders as their recreating parts.«[70] Gradić's conclusion cited above deserves an exhaustive commentary.

(1) It is significant from the methodological point of view. It demands the verification of Euclid's third axiom in a concrete geometrical proof. Thus, it is justifiable to ask for the origin and intention of the demand for verification of the deductive conclusion included in the third axiom. Evidently, the demand is based upon the fact that the major premise, i.e. the equality of the wholes preceding the subtraction of the parts, has by no means been proved. On the contrary, it is something that awaits to be proved.

(2) Gradić's demand for verification, in its very contents, insists on the logical link between Euclid's second and the third axioms, thus persisting on addition as a procedure inverse to subtraction. It should be noted that the problem of measuring a geometrical creation is recognized within Aristotelian heritage as a problem of continuum, expressed equally in two approaches: composition (*compositio continui*) and resolution (*resolutio continui*). Historical reasons herein present are not crucial. Gradić is determined in his understanding that within the considered problem, subtraction is not a finite procedure, so every subtracted part or parts must be taken into account as, according to the construction, there are infinitely many. From the standpoint of the second axiom, which uses the mathematical operation inverse to the one in the third axiom, it

[69] Gradius, »De loco Galilaei,« p. 51: »..., si, ut res est, nec punctum ad conum, nec circumferentia ad craterem tanquam partes integrales concurrunt;«

[70] Gradius, »De loco Galilaei,« p. 51: »nam ad verificandum axioma de aequalitate residui ex aequalitate ablatorum B totis aequalibus, necesse est, ut illa tot aequalia quae invicem comparantur B suo quaeque ablato, & residuo tanquam B partibus integrantibus componantur, ...«

means that, in the stated case, addition is also not a finite procedure leading to the demand of establishing the sum of the infinite series. Based on the difficulty of understanding this sum and its relation to the limit, Gradić formulates two opposing suppositions: limits *are* or *are not* component parts of the solid they limit. Therefore, Gradić's demand for verification of Euclid's axiom actualizes once again the problem of the sum of the infinite series.

As a result, it follows that Gradić considered the statement of Euclid's third axiom to have been a logical deduction applied only in geometrical proofs when: (1) it is already proven that the premises are true, and (2) the concept of operation that the statement of the axiom implies (i.e. the concept of subtraction of infinite parts) is defined in a mathematically precise way.

(3) Gradić's conclusion considers the recreation of the wholes which preceded subtraction. Such reconstruction is not possible, due to the initial hypothesis that the limits are not component parts of the solid. Logically speaking, there is no sense in involving something that is, according to the initial supposition separate from the whole, in the reconstruction of the whole. Thus, the verification process leads to the contradiction of the initial supposition.

(4) Gradić illustrates his concise conclusion with an example from seafaring life. It is an extremely successful comparison: »If some wheat were to be taken from a trireme and the same quantity from a small wherry, it could be concluded that the wherry is equal to the trireme.«[71] The craft loaded with wheat vividly illustrates the initial supposition on the limit which does not belong to the object it limits. The walls of the cargo space aboard the wherry and the trireme are certainly not made of wheat. According to »naive« understanding characteristic of everyday life, the trireme and the wherry carry an infinite number of grains of wheat. The discharge of these grains can be carried out in the same manner from both vessels by means of an equipment capable of unloading the equal quantity of grain, preferably a small one. Moreover, the discharge can be performed in a sufficiently large number of steps, so that someone attempting to count the exact number of steps could be easily confused. All in all, the counter could justly claim that after each step the equal quantity of grain is discharged. But that does not guarantee that the wherry and the trireme are vessels with equally large cargo space.

[71] Gradius, »De loco Galilaei,« p. 51: »alioquin si quis auferat ex aliqua triremi medium frumenti eandemque quantitatem ex aliquo parvo lintre concludere posset lintrem esse triremi aequalem.«

Gradić’s research into the relation between a geometrical object and its limit, carried out within a geometrical construction of Galileo’s paradox, refutes both initial suppositions. Point and circumference of the circle neither are or are not component parts of cone and crater. Contrary to Gradić’s expectations, it follows that the relation between geometrical object and its limit is undecidable for the establishment of error in Galileo’s paradox.

Throughout his treatise Gradić sought to reach mathematical results that could be expressed in fundamental topological categories. These attempts, in order of their appearance in the text, are:

(1) the immeasurability of a point;

(2) uniform procession as a transformation that safeguards the property of equality between sections of the cone and the crater, thereby defining this transformation by means of the property of equality;

(3) the relation of a geometrical object and its limit from the point of view of the dimension of geometrical quantities;

(4) the contraction of a surface toward a point and the contraction of a surface toward a circle as continuous transformations with essentially different results;

(5) the contraction of a line toward a point as a form of progression in a limiting process in which point is recognized as absolute progression, that is, as limit;

(6) the correspondence of elements that compose geometrical bodies as means of establishing an equal number of elements;

(7) the transition from a finite to an infinite number of elements that make up the relative volumes of the cone and the crater;

(8) the relation of a geometrical object and its limit, examined by means of suppositions that use the expressions »to be part of« and »not to be part of«, making the translation of these suppositions into contemporary set symbols fairly simple:

1. Fr A d A or Fr A 1 A ... 0,
2. Fr A 1 A = 0,

where A is a set and Fr A is the front or limit of set A;

(9) the coalescence of a geometrical object by its limit.

Although Gradić's cited approaches were never brought to the level of rounded mathematical theory, most of which failed to yield the desired results, in some cases missing the chance of fruitful generalization, nevertheless, one sees here an active mathematical approach owing to which Stjepan Gradić must be counted among the pioneers of *topological thought* in mathematics.

6. The originality of Gradić's approach in comparison with those of Cavalieri and Fabri

The study of Gradić's treatise *De loco Galilaei* and the source writings of his professor Cavalieri as well as his collocutor Fabri from the Roman scientific milieu, enable us to make a thorough investigation into Gradić's originality and as to what extent he developed or inherited the ideas of Cavalieri and Fabri.

In the analysis of Galileo's paradox, Gradić concentrates on three questions:

(1) What is the meaning of the final consequence of the paradox?

(2) What is the character of uniform procession by which the plane sections of the cone and the crater approach the tops of these solids?

(3) How should a method for measuring volumes with curved limits be built?

According to Gradić, to claim the equality of a point and a line is, in final consequence, to state that a nondimensional point is equal to a three-dimensional quantity, and therefore contradicts the idea of Euclid's definition of the point. But whereas Euclid's definition of point includes the concept of indivisibility (*cujus pars nulla est*), Gradić prefers to characterize it as immeasurable (*res nullius mensurae*).[72] Therefore, Gradić's terminological choice implicitly disputes the indivisible, especially if compared with Cavalieri's or Fabri's explicit citing of the method of indivisibles in the analysis of Galileo's paradox of the bowl. Cavalieri, however, in the described corollary of theorem 5, examines the ratios of geometrical figures based on the supposition that they are

[72] Gradius, »De loco Galilaei,« p. 42: »... punctum ipsum, hoc est rem nullius mensurae ...«

composed of lines, and Fabri deduces the equality of the cone and the round razor from the equality of the sum of all the circles and the sum of all the circular bands these figures resolve into.

Gradić's second terminological choice is even more far-reaching: *uniformis processio*. This new term designates the procedure applied to the bases of the cone and the crater when understood as sections of a plane which continuously approaches the top of the cone and the circumference of the crater respectively. Thus, the equality of the circle and the circular band as two plane sections is dynamically understood. Examining the character of uniform procession means stating the reason or reasons for the appearance of the paradox where the point equals the line at the end of the procedure. Gradić's first insight refers to the limits of equal surfaces. Gradić explicitly states that the equality of surfaces does not necessarily result in the equality of their limits. Discrepancy in the magnitude of the limit of the circle and the circular band is evident for the arbitrarily chosen plane of the section and even further emphasized with the progress of procession (*in progressu processionis*).[73] Gradić further adds that in geometry there are no rules by which the equality of limits could be deduced from the equality of quantities limited by those limits.[74] Moreover, Gradić's term *uniformis processio* for the intersecting plane includes, as its full meaning, approaching the limit. Obviously, Gradić examines the problem of limits of equal surfaces in an essentially different manner from Cavalieri. Within Cavalieri's method of indivisibles, particularly presented in his correspondence with Galileo, the problem of the equality of surfaces does not at the same time include the problem of their limits, due to the reason that the idea of the surface of a geometrical figure as the sum of all lines of the figure excludes the limit of the figure.

Gradić mostly concentrated on the proof of the equality of cone and crater presented by Luca Valerio in his work *De centro gravitatis solidorum*. Whereas Galileo cites Valerio's proof, Gradić makes an elaborate study with the intent of finding the source of the paradox during the proof procedure. Reviving the Archimedean method for determining the volume of solids with curved surfaces, Valerio divided the cone and the crater by means of equidistant planes, parallel to the common base of those two solids. In Gradić's terminological

[73] Gradius, »De loco Galilaei,« p. 47.

[74] Gradius, »De loco Galilaei,« pp. 44-45.

description, Valerio acquired cylinders and cylindrical orbs (*orbes cylindrici*), forming indented forms unequal to the volume of cone and crater, but approaching them by volume as the number of equidistant planes intersecting both solids increases. Gradić's statement stood well out from the scientific milieu of his time by the *rigor* of its mathematical expressions.[75] As an example, Gradić uses the following expressions: »every singular cylinder, component of the cone, corresponds to a cylindrical orb of equal magnitude, [component of the crater]«, »if such a multiplication of sections [by plane] increased to a number of any magnitude«, »if that indented figure composed of any number of cylindrical orbs approaches any approximation of the crater«. These expressions enable Gradić to apply the *original* description of the transition from a finite amount to an infinite amount of parts of the cone and the crater, i.e. the description of the transition from indented forms toward solids whose surface is quite (*plane*) conical or perfectly (*perfecte*) crater-like. Besides, Gradić introduces the concept of an arbitrarily small approximation (*quaecumque proximitas*) and concludes that the approximations observed simultaneously in relation to cone and crater remain constantly equal during the described process. The shortcoming of Gradić's approach is in his actual and not potential understanding of the sum of the infinite series and the error there resulting as the inscribed denticulate form is considered instead of the whole solid.

Gradić's understanding of Valerio's proof is characterized by another of his ideas. Here Gradić repeatedly questions the relation between geometrical form and its limit, this time with the aid of Euclid's axioms (A2-A4) on the equality and inequality of quantities.[76] As Gradić understands it, two contrary suppositions appear. According to the first, the point and the circumference are component parts of the cone and the crater; according to the second, the point and the circumference are not component parts. Compared with Cavalieri's approach, Gradić's first supposition is in direct contradiction with the method of indivisibles, while Gradić's second supposition inherits the same starting point as the method of indivisibles. From such a standpoint we can also revaluate Gradić's conclusion that the suppositions lead to contradiction. Gradić's study of the relation between a geometrical object and its limit suggests that the

[75] See note 65.

[76] See note 67.

starting-point of the method of indivisibles is not decisive for the establishment of the source of Galileo's paradox of the bowl. Moreover, Gradić emphasizes that the second supposition as the starting-point of the method of indivisibles does not enable the verification of Euclid's third axiom.[77] As it is supposed that the point and the circumference of the circle are not component parts of cone and crater, taken as the starting-point by Cavalieri, this means that Gradić has located a weak spot of the method of indivisibles.

Gradić persisted in his mathematical elaboration throughout his approach and evaluation of Galileo's paradox. For this purpose, he applied mathematical concepts such as *mensura*, *uniformis processio* and *proximitas*, formulated strict mathematical expressions, and offered new ideas on how to understand the approach to the limit of a geometric form as well as the relation between the geometric form and its limit. While Fabri, confronted with the same problem, reached for metaphysical argumentation, Gradić resisted that temptation in spite of his tremendous knowledge in philosophy.

Gradić's approach to Galileo's paradox of the bowl does not only represent a separate view in the 1660s, but an original understanding of the limiting process. The comparative study of sources in which Gradić's predecessors Cavalieri and Fabri proved or attempted to prove the contradictory conclusion on the equality of point and line, only encourages the estimation that Gradić, in his study of the limiting process, introduced an original approach to the infinitesimal method. Unfortunately, his work was ignored and he did not influence the development of mathematics in the seventeenth century and later. Moreover, Ruđer Bošković, the professor of mathematics at the Collegium Romanum, where a century before Stjepan Gradić had been an alumnus, was not familiar with Gradić's mathematical work either. Bošković considered Gradić more an author of elegant Latin verse and a censor of literary production in Italy than a mathematician.[78]

[77] See note 70.

[78] Rogerius Josephus Boscovich, *De Solis ac Lunae defectibus* (Venetiis: Typis Antonii Zatta, 1761), note on p. 261: »Stephanum Gradium elegantissimum latinum poetam eo ipso superiore saeculo, quo usque adeo conciderant in Italia humaniores litterae corruptae penitus, ac depravatae;«

7. Conclusion

Gradić's evaluation of Galileo's paradox of the bowl introduces a variety of mathematical contents. It includes:

(1) implicite criticism of the concept of the indivisible;

(2) the idea that contraction of a line to a point in the field of geometry and mechanics represents an identical procedure, i.e. the approach to a limit, although this concept is based on a qualitative geometrical description without the aid of mathematical means that would transform this exact understanding of the limiting process into a new mathematical method;

(3) a revaluation of the axiomatic base of Euclid's geometry;

(4) the development of topological ideas, especially within study of the relation between a geometric object and its limit.

The comparison between Gradić's approach and those of his contemporaries, his professor Cavalieri and his correspondent Fabri, who use the method of indivisibles in their analyses of Galileo's paradox, only proves that Gradić, in his study of the approach to a limit, paved the original way to the infinitesimal method.

APPENDIX

Stjepan Gradić

On Luca Valerio's proof for the equality of the cone and the crater

Non defuit qui hunc argumentandi modum B Salviato usurpatum sustineri posse diceret exemplo ratiocinationis, qua in hac materia utitur Lucas Valerius in suo tractatu de Centro gravitatis B Salviato laudatus; ibi enim auctor ille gravissimus aequalitatem inter craterem & conum demonstrat ex eo quod per quaedam plana basi cylindri aequidistantia &c. dividitur conus quidem in plures cylindros, crater vero in totidem orbes cylindricos (voco orbes cylindricos solidum illud quod restat ex majore cylindro, si minor cylindrus ejusdem axis ab eo auferatur) ita ut unicuique cylindro componenti conum respondeat orbis cylindricus ejusdem magnitudinis, recte autem ex aequalitate singularum partium aequalitas consurgit universarum, & sic ex aequalitate cylindrorum, & orbium cylindricorum aequalitas crateris & coni prout in elementis. Etsi autem ordinatis per sectiones, quas diximus, dictis cylindris ex una parte, & orbibus ex altera, nec omnes ii cylindri integrum conum nec omnes orbes integram lancem efficiunt, ex quo ex hujusmodi orbium cylindricorum itemque cylindrorum aggregatis superficies nec in cono plane conica, nec in cratere perfecte, ut ita dicam, craterica coalescat, sed interpositis utrique quibusdam frustulis evadunt quodammodo denticulatae, perinde tamen est ac si talia frusta minime intervenirent, quandoquidem talis aequalitas singulorum cylindrorum & orbium semper procedit, etiamsi ad quemcumque numerum talis multiplicatio sectionum excrescat, sequaturque id, quod necessario sequi debet, tanta diminutio singulorum hujusmodi frustrorum, ut omnia simul posita minorem efficiant quacumque data quantitate magnitudinem; hac enim ratione non potest unquam accidere, ut figura illa denticulata composita ex quotcumque orbibus cylindricis, & sic ad quamcumque proximitatem crateris admota non aequalis sit alteri figurae pariter denticulatae, compositae ex totidem cylindris, & ad coni aequalitatem pari proximitate accedentis.

Stephanus Gradius, »Dissertatio III. De loco Galilaei, quo punctum lineae aequale pronuntiat,« in Gradius, *Dissertationes physico-mathematicae quatuor* (Amstelodami: Apud Danielem Elsevirium, 1680), pp. 39-54, on pp. 47-50.